Solution to Assignment 1

1. Show that the function $\varphi(x) = 1/x, x \in (0,1]$, and $\varphi(0) = 1$ is not integrable on $[0,1]$.

   Solution Suppose on the contrary that φ is integrable. For all partitions with small norm $\|P\|$, their associated Riemann sums should come close to the same number

   $$I = \int_0^1 \varphi$$

   regardless of the tags chosen. However, consider an arbitrary partition with tags $\{z_j\}$. The Riemann sum

   $$\begin{aligned} S(\varphi, P) &= \sum_{j=1}^{n} \varphi(z_j)\Delta x_j \\ &= \frac{1}{z_1}\Delta x_1 + \sum_{j=2}^{n} \frac{1}{z_j}\Delta x_j \\ &\geq \frac{1}{z_1}\Delta x_1 \ . \end{aligned}$$

   While $z_j, j \geq 2$ are fixed, if we let z_1 becomes very small, $1/z_1$ becomes very large, so $S(\varphi, P)$ could become arbitrarily large and cannot come close to I. Therefore, φ cannot be integrable.

   Note In fact, it can be shown that all unbounded functions are non-integrable.

2. Consider the function g in $\mathbb{R}^2$ defined by $g(x,y) = 1$ whenever x, y are rational numbers and equals to 0 otherwise. Show that g is not integrable in any rectangle.

   Solution Let P be any partition of the rectangle. By choosing tags points $p_{ij}(x_i^*, y_j^*)$ where x_i^* and y_j^* are rational numbers,

   $$\sum_{i,j} g(x_i^*, y_j^*)\Delta x_i \Delta y_j = \sum_{i,j} \Delta x_i \Delta y_j$$

   which is equal to the area of R. On the other hand, by choosing the tags so that x_i^* is irrational, $g(x_i^*, y_j^*) = 0$ so that

   $$\sum_{i,j} g(x_i^*, y_j^*)\Delta x_i \Delta y_j = \sum_{i,j} 0 \times \Delta x_i \Delta y_j = 0 \ .$$

   Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit no matter how small their norms are. We conclude that g is not integrable.